

On the Sendov conjecture and the critical points of polynomials *

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In this paper, we obtain new results on the critical points of a polynomial. We discuss the Sendov conjecture for polynomials of degree nine.

Keywords: critical points, extremal polynomial, derivative.

1 Introduction

Let \mathcal{P}_n denote the set of all monic polynomials of degree $n(\geq 2)$ of the form

$$p(z) = \prod_{k=1}^n (z - z_k), \quad |z_k| \leq 1 (k = 1, \dots, n)$$

with

$$p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j), \quad |\zeta_j| \leq 1 (j = 1, \dots, n-1).$$

Write $I(z_k) = \min_{1 \leq j \leq n-1} |z_k - \zeta_j|$, $I(p) = \max_{1 \leq k \leq n} I(z_k)$, and $I(\mathcal{P}_n) = \sup_{p \in \mathcal{P}_n} I(p)$.

It was showed that there exists an extremal polynomial p_n^* , i.e., $I(\mathcal{P}_n) = I(p_n^*)$ and that p_n^* has at least one zero on each subarc of the unit circle of length π (see [3], [9]).

It will suffice to prove the Sendov conjecture assuming p is an extremal polynomial of the following form,

$$p(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k), \quad |z_k| \leq 1 (k = 1, \dots, n-1)$$

with

$$p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j), \quad |\zeta_j| \leq 1 (j = 1, \dots, n-1), \quad a \in [0, 1].$$

Let $r_k = |a - z_k|$, $\rho_j = |a - \zeta_j|$ for $k, j = 1, 2, \dots, n-1$. By relabeling we suppose that

$$\rho_1 \leq \rho_2 \leq \dots \leq \rho_{n-1}, \quad r_1 \leq r_2 \leq \dots \leq r_{n-1}. \quad (1.1)$$

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We have (see [5],[7],[10])

$$2\rho_1 \sin\left(\frac{\pi}{n}\right) \leq r_k \leq 1 + a, \quad k = 1, 2, \dots, n-1. \quad (1.2)$$

Sendov conjecture. The disk $|z - a| \leq 1$ contains a zero of $p'(z)$.

If $a = 0$ or $a = 1$, Sendov conjecture is true (see [7],[11]), we suppose $a \in (0, 1)$.

In this paper, we obtain the following theorems.

Theorem 1. If $1 - (1 - |p(0)|)^{\frac{1}{n}} \leq \lambda \leq \sin(\frac{\pi}{n})$ and $\lambda < a$, $\rho_1 \geq 1$, then there exists a critical point $\zeta_0 = a + \rho_0 e^{i\theta_0}$ such that $\operatorname{Re} \zeta_0 \geq \frac{1}{2}(a - \frac{\lambda(\lambda+2)}{a})$.

This theorem improves the previous known results (see [2],[4],[5],[8]).

Theorem 2. If p is an extremal polynomial, $n = 9$, $p(a) = 0$, and $a \in [0, 0.845)$, then the disk $|z - a| \leq 1$ contains a zero of $p'(z)$.

Theorem 3. For $\rho > 0$, and m is real number, we have

$$\prod_{r_k \geq \rho} r_k \rho^{-1} \leq \prod_{\rho_j \geq \rho^m} \rho_j \rho^{-m} \prod_{\rho^{m-1} 2 \sin \frac{\pi k}{n} \geq 1} \rho^{m-1} 2 \sin \frac{\pi k}{n}.$$

2 Proof of the Theorem 1

Lemma 2.1. If $0 < a < 1$ and $\rho_1 \geq 1$, then

$$|p(z)| > 1 - (1 - \lambda)^n, \text{ for } 0 < |z - a| = \lambda \leq \sin(\pi/n).$$

This is Lemma 2.16 of [2] (see [1]).

Proof of the Theorem 1. We apply Lemma 2.1 to conclude that

$$|p(z)| > 1 - (1 - \lambda)^n \geq |p(0)|, \quad 0 < |z - a| = \lambda.$$

Since $p(z)$ is univalent in $|z - a| \leq \lambda$, it follows that there exists a unique point z_0 with $|z_0 - a| < \lambda$ such that $p(0) = p(z_0)$. We assume that $\operatorname{Im} z_0 \geq 0$ (if not, consider $\overline{p(\bar{z})}$).

By a variant of the Grace-Heawood theorem, there exists a critical point in each of the half-planes bounded by the perpendicular bisector L of the segment from 0 to z_0 . Let

$\zeta_0 = a + \rho_0 e^{i\theta_0}$ be the critical point in the half-plane containing z_0 .

The equation of L is $|z| = |z - z_0|$, that is

$$z\bar{z} = (z - z_0)(\bar{z} - \bar{z}_0), \quad (2.1)$$

then

$$|z_0|^2 = z\overline{z_0} + \overline{z}z_0. \quad (2.2)$$

Let $z^* = e^{i\beta_0}$ be the joint point of L and the circle $|z| = 1$, $\operatorname{Im} z^* \geq 0$, then

$$|z_0|^2 = e^{i\beta_0}\overline{z_0} + e^{-i\beta_0}z_0.$$

Hence

$$e^{i\beta_0} = \frac{z_0}{2} \pm \frac{z_0}{2|z_0|} \sqrt{|z_0|^2 - 4},$$

that is

$$e^{i\beta_0} = \left(\frac{1}{2} \pm \frac{i}{2|z_0|} \sqrt{4 - |z_0|^2}\right) z_0. \quad (2.3)$$

If $z_0 = a$, the theorem is true, we write $z_0 = a + re^{i\alpha}$, $0 < r < \lambda$. We choose

$$\begin{aligned} \cos \beta_0 &= \frac{1}{2}(a + r \cos \alpha) - \frac{1}{2|z_0|} \sqrt{4 - |z_0|^2} r \sin \alpha, \\ \cos \beta_0 &= \frac{1}{2}(a + r \cos \alpha) - \frac{1}{2} \frac{\sqrt{4 - a^2 - 2ar \cos \alpha - r^2}}{\sqrt{a^2 + 2ar \cos \alpha + r^2}} r \sin \alpha. \end{aligned} \quad (2.4)$$

We fix r and consider the circle $|z - a| = r$, let $\sin \alpha_1 = \frac{\sqrt{a^2 - r^2}}{a}$, $x = \cos \alpha$ and $F(x) =$

$$x - \sqrt{\frac{4 - a^2 - 2arx - r^2}{a^2 + 2arx + r^2}} \sqrt{1 - x^2},$$

then

$$\cos \beta_0 = \frac{1}{2}(a + rF(x)), \quad (2.5)$$

it is sufficient to give lower bound of $F(x)$ for $x \in [-1, -\frac{r}{a}]$.

Write $G(x) = -F(-x)$, then

$$G(x) = x + \sqrt{\frac{4 - a^2 + 2arx - r^2}{a^2 - 2arx + r^2}} \sqrt{1 - x^2}, \quad (2.6)$$

it is sufficient to give upper bound of $G(x)$ for $x \in [\frac{r}{a}, 1]$.

Write $\psi = \frac{4 - a^2 + 2arx - r^2}{a^2 - 2arx + r^2}$, $\phi = a^2 - 2arx + r^2$.

We have

$$G'(x) = 1 + \frac{1}{2} \psi^{-\frac{1}{2}} \psi' \sqrt{1 - x^2} - \psi^{\frac{1}{2}} x (1 - x^2)^{-\frac{1}{2}}, \quad (2.7)$$

and $\psi' = 8ar(a^2 - 2arx + r^2)^{-2}$,

hence

$$G'(x) = \phi^{-2}\psi^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}G_1, \quad (2.8)$$

where

$$G_1 = \phi^{\frac{3}{2}}(4-\phi)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} + 4ar(1-x^2) - x\phi(4-\phi). \quad (2.9)$$

We have $x = \frac{a^2+r^2-\phi}{2ar}$, then

$$G_1 = \frac{1}{2ar}G_2, \quad (2.10)$$

where

$$\begin{aligned} G_2 &= \phi^{\frac{3}{2}}(4-\phi)^{\frac{1}{2}}(4a^2r^2 - (a^2+r^2)^2 + 2(a^2+r^2)\phi - \phi^2)^{\frac{1}{2}} \\ &+ 8a^2r^2 - 2(a^2+r^2)^2 + 4(a^2+r^2)\phi - 2\phi^2 - (a^2+r^2-\phi)\phi(4-\phi), \end{aligned}$$

hence

$$\begin{aligned} G_2 &= \phi^{\frac{3}{2}}(4-\phi)^{\frac{1}{2}}(4a^2r^2 - (a^2+r^2)^2 + 2(a^2+r^2)\phi - \phi^2)^{\frac{1}{2}} \\ &+ 8a^2r^2 - 2(a^2+r^2)^2 + (a^2+r^2+2)\phi^2 - \phi^3, \end{aligned}$$

and $\phi \in [(a-r)^2, a^2-r^2]$.

The roots of G_2 satisfy

$$\begin{aligned} L &= \phi^3(4-\phi)(4a^2r^2 - (a^2+r^2)^2 + 2(a^2+r^2)\phi - \phi^2) = \\ &(\phi^3 - (a^2+r^2+2)\phi^2 + 2(a^2+r^2)^2 - 8a^2r^2)^2 = R, \end{aligned}$$

say.

We have

$$\begin{aligned} L &= \phi^6 - 2(a^2+r^2+2)\phi^5 + (8(a^2+r^2) - 4a^2r^2 + (a^2+r^2)^2)\phi^4 + (16a^2r^2 - 4(a^2+r^2)^2)\phi^3, \\ R &= \phi^6 - 2(a^2+r^2+2)\phi^5 + (a^2+r^2+2)^2\phi^4 + 4(a^2-r^2)^2\phi^3 - 4(a^2+r^2+2)(a^2-r^2)^2\phi^2 + \\ &4(a^2-r^2)^4. \end{aligned}$$

Let $d = 1 - r^2$, $e_1 = a^2 - 1$, by $L = R$, we deduce

$$e_1d\phi^4 - 2(e_1+d)^2\phi^3 + (4+e_1-d)(e_1+d)^2\phi^2 - (e_1+d)^4 = 0,$$

that is

$$(e_1\phi^2 - 2(e_1 + d)\phi - (e_1 + d)^2)(d\phi^2 - 2(e_1 + d)\phi + (e_1 + d)^2) = 0,$$

there is only the root $\phi_0 = \frac{a^2 - r^2}{1+r} \in [(a-r)^2, a^2 - r^2]$, and

$$x_0 = \frac{a^2 + r^2 - \phi_0}{2ar} = \frac{2r + a^2 + r^2}{2a(1+r)} \in [\frac{r}{a}, 1].$$

We have $G'(1-0) < 0$, $G'(\frac{r}{a}) > 0$, $G(x_0)$ is the maxima value of $G(x)$ for $x \in [\frac{r}{a}, 1]$.

We obtain $G(x_0) = \frac{r+2}{a}$, and $G(x) \leq \frac{r+2}{a}$, $x \in [\frac{r}{a}, 1]$, hence

$$F(x) \geq -\frac{r+2}{a}, \quad x \in [-1, -\frac{r}{a}],$$

$$\cos \beta_0 \geq \frac{1}{2}(a - \frac{r(r+2)}{a}) \geq \frac{1}{2}(a - \frac{\lambda(\lambda+2)}{a}),$$

by the Gauss-Lucas theorem, Theorem 1 follows.

3 Proof of the Theorem 2

We improve the methods of [5].

If $p(0) = 0$, the Sendov conjecture is true (see Satz 3 of [11]).

In this section, we assume that $z_k \neq 0$, $z_k \neq a$, $k = 1, \dots, n-1$ and that $p(z)$ is extremal: $I(\mathcal{P}_n) = I(p) = I(a) = \rho_1$.

Lemma 3.1. If $c_k (k = 1, \dots, N)$, m , M , C are positive constants with

$$m \leq c_k \leq M, \quad \prod_{k=1}^N c_k \geq C \text{ and } m^N \leq C \leq M^N, \text{ then}$$

$$\sum_{k=1}^N \frac{1}{c_k^2} \leq \frac{N-v}{m^2} + \frac{v-1}{M^2} + \left\{ \frac{m^{N-v} M^{v-1}}{C} \right\}^2,$$

where $v = \min\{j \in \mathbb{Z} : M^j m^{N-j} \geq C\}$.

Proof. See Lemma 7.3.9 of [10].

Lemma 3.2. If $w \neq a$, then

$$\prod_{k=1}^{n-1} (a - z_k) = n \prod_{j=1}^{n-1} (a - \zeta_j),$$

$$\prod_{k=1}^{n-1} r_k = n \prod_{j=1}^{n-1} \rho_j,$$

$$\sum_{j=1}^{n-1} \frac{1}{a - \zeta_j} = \sum_{k=1}^{n-1} \frac{2}{a - z_k},$$

$$\operatorname{Re}\left\{\frac{1}{a-w}\right\} = \frac{1}{2a} - \frac{|w|^2 - a^2}{2a|a-w|^2},$$

$$\frac{1}{n}\left(a + \sum_{k=1}^{n-1} z_k\right) = \frac{1}{n-1} \sum_{j=1}^{n-1} \zeta_j.$$

Proof. See (2.2), (2.3), (2.5) of [5], and (2.3.1) of [10] .

Let

$$\gamma_j = \frac{\zeta_j - a}{a\zeta_j - 1} \text{ and } w_k = \frac{z_k - a}{az_k - 1}.$$

By (2.11) of [5], we have

$$\prod_{j=1}^{n-1} |\gamma_j| \leq \frac{\prod_{k=1}^{n-1} |w_k|}{n - \frac{4a^2}{1+a^2} - a \sum_{k=1}^{n-3} \operatorname{Re} w_k}.$$

We take $n = 9$, then

$$\prod_{j=1}^8 |\gamma_j| \leq \frac{\prod_{k=1}^8 |w_k|}{9 - 4a^2/(1+a^2) - 6a}. \quad (3.1)$$

Lemma 3.3. Let $A_9 (A_9 = 0.4314 \dots)$ be the smallest positive root of $9 - \frac{4x^2}{1+x^2} - 6x - (1+x-x^2)^8 = 0$, and $a \leq A_9$, then $\rho_1 \leq 1$.

Proof. This is Lemma 3.2 of [5].

Lemma 3.4. If $\rho_1 > 1$, and $\zeta_0 = a + \rho_0 e^{i\theta_0}$ is the critical point in Theorem 1, $\gamma_0 = \frac{\zeta_0 - a}{a\zeta_0 - 1}$, then

$$|\gamma_0| > \frac{1}{\sqrt{1 + \lambda(\lambda+2) - a^2\lambda(\lambda+2)}}.$$

Proof. We have $|\gamma_0|^2 = \frac{\rho_0^2}{a^2|\zeta_0|^2 - 2a\operatorname{Re}\zeta_0 + 1}$, $\operatorname{Re}\zeta_0 \geq \frac{1}{2}\left(a - \frac{\lambda(\lambda+2)}{a}\right)$, hence

$$\rho_0 \cos \theta_0 \geq -\frac{1}{2}\left(a + \frac{\lambda(\lambda+2)}{a}\right), \text{ and}$$

$$|\gamma_0|^2 = \frac{\rho_0^2}{a^4 - 2a^2 + 1 + a^2\rho_0^2 + 2a(a^2 - 1)\rho_0 \cos \theta_0} \geq \frac{\rho_0^2}{a^4 - 2a^2 + 1 + a^2\rho_0^2 + (1 - a^2)(a^2 + \lambda(\lambda+2))}$$

$$> \frac{1}{1 + \lambda(\lambda+2) - a^2\lambda(\lambda+2)},$$

the lemma follows.

Lemma 3.5. If $|\gamma_j| \leq \frac{1}{1+a-a^2}$, then $\rho_j \leq 1$.

Proof. See Lemma 1 of [4].

Lemma 3.6. If $R \in (0, 1]$, $w = \frac{z-a}{az-1}$, $|z| \leq R$, then

$$|w| \leq \frac{|z| + a}{a|z| + 1} \leq \frac{R + a}{aR + 1}.$$

Proof. We have

$$|w|^2 = \frac{|z-a|^2}{|az-1|^2} = \frac{|z|^2 - 2a\operatorname{Re}z + a^2}{a^2|z|^2 - 2a\operatorname{Re}z + 1} \leq \frac{|z|^2 + 2a|z| + a^2}{a^2|z|^2 + 2a|z| + 1} = \frac{(|z|+a)^2}{(a|z|+1)^2},$$

hence

$$|w| \leq \frac{|z|+a}{a|z|+1} \leq \frac{R+a}{aR+1}.$$

Write $B = B(R) = \frac{R+a}{aR+1}$.

Lemma 3.7. If $a \in [0.4314, 0.51952]$, or if there exists $|z_k| \leq 0.4$ for $a \in [0.5195, 1]$, then

$$I(a) = \rho_1 \leq 1.$$

Proof. If $I(a) > 1$ and there exists $|z_k| \leq R$, then, by (3.1), Lemma 3.4, and Lemma 3.5,

there exists some γ_{j_0} ,

$$\frac{|\gamma_{j_0}|^7}{\sqrt{1+\lambda(\lambda+2)-a^2\lambda(\lambda+2)}} < \prod_{j=1}^8 |\gamma_j| < \frac{B}{9-4a^2/(1+a^2)-6a},$$

hence

$$|\gamma_{j_0}|^7 < \frac{\sqrt{1+\lambda(\lambda+2)-a^2\lambda(\lambda+2)}}{9-4a^2/(1+a^2)-6a} B.$$

By Lemma 3.5, it suffices to show

$$(9 - 4a^2/(1 + a^2) - 6a) \frac{aR + 1}{R + a} - \sqrt{1 + (1 - a^2)\lambda(\lambda + 2)}(1 + a - a^2)^7 \geq 0. \quad (3.2)$$

We consider the conditions for λ ,

$$|p(0)| = a \prod_{k=1}^8 |z_k| \leq aR, \quad 1 - (1 - |p(0)|)^{\frac{1}{9}} \leq \lambda \leq \sin(\frac{\pi}{9}),$$

we choose

$$\lambda = 1 - (1 - aR)^{\frac{1}{9}},$$

and R satisfies $R \leq a^{-1}(1 - (1 - \sin(\frac{\pi}{9}))^9)$.

If $a \in [0.4314, 0.51952]$, we take $R = 1$, then

$$9 - 4a^2/(1 + a^2) - 6a - (1 + a - a^2)^7 \sqrt{1 + (1 - a^2)\lambda(\lambda + 2)} > 0,$$

we obtain (3.2).

If $a \in [0.5195, 1]$, and there exists $|z_k| \leq 0.4$, we take $R = 0.4$, (3.2) holds again, the lemma follows.

Lemma 3.8. If $a \in (0.5195, 1]$, $x \in [0.4, 1]$, then

$$x^2 + \frac{(1-x^2)(a^2+x^2)^2}{9x^2} \leq 1.$$

Proof. It is sufficient to show

$$(1+x^2)^2 \leq 9x^2,$$

this holds for $x \in [0.4, 1]$, the lemma follows.

We have $\frac{p'(0)}{p(0)} = -(\frac{1}{a} + \sum_{k=1}^8 \frac{1}{z_k})$, and $9 \prod_{j=1}^8 |\zeta_j| = (a \prod_{k=1}^8 |z_k|)|\frac{1}{a} + \sum_{k=1}^8 \frac{1}{z_k}|$.

let $\Delta = \operatorname{Re}(\frac{1}{a} + \sum_{k=1}^8 \frac{1}{z_k})$, $\sigma = \sum_{k=1}^8 \frac{1}{r_k^2}$, then $9 \prod_{j=1}^8 |\zeta_j| \geq -\Delta a \prod_{k=1}^8 |z_k|$.

Lemma 3.9. If $\rho_1 > 1$, $a \in [0.5195, 1]$, and $|z_k| \in [0.4, 1]$, $k = 1, \dots, 8$, then

$$\Delta \leq -\frac{8}{a} + 8a + \frac{9}{8a}(1-a^2)\sigma.$$

Proof. Write $z_k = |z_k|e^{i\theta_k}$, $\zeta_j = a + \rho_j e^{it_j}$, $1 \leq k, j \leq 8$. By Lemma 3.2, we have

$$\begin{aligned} \Delta &= \frac{1}{a} + \operatorname{Re} \sum_{k=1}^8 z_k + \sum_{k=1}^8 \frac{1-|z_k|^2}{|z_k|} \cos \theta_k, \\ \Delta &= \frac{1-a^2}{a} + \frac{9}{8} \operatorname{Re} \sum_{j=1}^8 \zeta_j + \sum_{k=1}^8 \frac{1-|z_k|^2}{|z_k|} \cos \theta_k, \\ \sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2} &= 8 + a \sum_{j=1}^8 \frac{\cos t_j}{\rho_j}. \end{aligned} \tag{3.3}$$

Since $\rho_j > 1$, we deduce $\cos t_j \leq 0$,

$$\sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2} \geq 8 + a \sum_{j=1}^8 \cos t_j.$$

But

$$\operatorname{Re} \sum_{j=1}^8 \zeta_j = 8a + \sum_{j=1}^8 \rho_j \cos t_j \leq 8a + \sum_{j=1}^8 \cos t_j,$$

hence

$$\operatorname{Re} \sum_{j=1}^8 \zeta_j \leq 8a - \frac{8}{a} + \frac{1}{a} \sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2}, \text{ and by (3.3)}$$

$$\Delta \leq 8a - \frac{8}{a} + \sum_{k=1}^8 \left(\frac{9}{8a} \frac{|z_k|^2 - a^2}{r_k^2} + \frac{1-|z_k|^2}{|z_k|} \cos \theta_k \right). \tag{3.4}$$

We want to prove

$$\frac{9}{8a} \frac{|z_k|^2 - a^2}{r_k^2} + \frac{1 - |z_k|^2}{|z_k|} \cos \theta_k \leq \frac{9}{8a} \frac{1 - a^2}{r_k^2}. \quad (3.5)$$

If $\cos \theta_k \leq 0$, (3.5) is valid, we assume $\cos \theta_k > 0$ and write $x = |z_k|$, $\theta = \theta_k$, then $r_k^2 = a^2 + x^2 - 2ax \cos \theta$. We want to show

$$x^2 + \frac{8a(1 - x^2)}{9x} (a^2 + x^2 - 2ax \cos \theta) \cos \theta \leq 1,$$

it suffices to prove

$$x^2 + \frac{(1 - x^2)(a^2 + x^2)^2}{9x^2} \leq 1,$$

for $a \in [0.5195, 1]$, $x \in [0.4, 1]$, this is true by Lemma 3.8, the lemma follows.

Lemma 3.10. Let $m = \frac{1}{4}$, $a \in [0.5, 0.95]$, $f(x) = \frac{x^2 - 1}{(1 - x^m)(a + x)^2}$, then $f'(x) > 0$, for $x \in [0.4, 1)$.

Proof. We have $f'(x) = \frac{a+x}{(1-x^m)^2(a+x)^4} Y$, where

$$Y = ((m - 2)x^{m+1} - mx^{m-1} + 2x)a + mx^{m+2} - (2 + m)x^m + 2.$$

If $Y = 0$, we will obtain $a > 0.955$ for $x \in [0.4, 1)$, hence $Y \neq 0$ for $a \in [0.5, 0.95]$, $x \in [0.4, 1)$. When $a = x = 0.8$, $Y > 0$, the lemma follows.

Lemma 3.11. If $\rho_1 \geq 1$, we have

$$\prod_{j=1}^8 |\zeta_j| \leq \left(\prod_{j=1}^8 \rho_j \right) (a^2 - 1 + \frac{1}{4} \sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2})^4.$$

Proof. By Lemma 3.2, we obtain $\sum_{j=1}^8 \frac{a^2 - |\zeta_j|^2}{\rho_j^2} = 8 + 2 \sum_{k=1}^8 \frac{a^2 - |z_k|^2}{r_k^2}$, then

$$\sum_{j=1}^8 \frac{|\zeta_j|^2}{\rho_j^2} \leq 8a^2 - 8 + 2 \sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2}.$$

Apply the arithmetic-geometric means inequality:

$$\prod_{j=1}^8 \frac{|\zeta_j|}{\rho_j} = \left(\prod_{j=1}^8 \frac{|\zeta_j|^2}{\rho_j^2} \right)^{\frac{1}{2}} \leq \left(\frac{1}{8} \sum_{j=1}^8 \frac{|\zeta_j|^2}{\rho_j^2} \right)^4,$$

the lemma follows.

We will use the following conditions

$$\frac{x^2 - 1}{(1 - x^m)(a + x)^2} + (1 - a^2)(\sigma - 4) \leq 0, \quad x \in [0.4, 1]. \quad (3.6)$$

Lemma 3.12. If $\sigma > 4$, $\rho_1 > 1$, and condition (3.6) holds with $m = \frac{1}{4}$, then

$$4^4(8 - \frac{9}{8}\sigma)(1 - a^2)^{-3} \leq (\sigma - 4)^4 9 \prod_{j=1}^8 \rho_j.$$

Proof. By Lemma 3.9 and Lemma 3.11,

$$(8 - 8a^2 - \frac{9}{8}(1 - a^2)\sigma) \prod_{k=1}^8 |z_k| \leq 9(\prod_{j=1}^8 \rho_j)(a^2 - 1 + \frac{1}{4} \sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2})^4.$$

Write

$$\Phi = (a^2 - 1 + \frac{1}{4} \sum_{k=1}^8 \frac{|z_k|^2 - a^2}{r_k^2})(\prod_{k=1}^8 |z_k|)^{-\frac{1}{4}},$$

then

$$8 - 8a^2 - \frac{9}{8}(1 - a^2)\sigma \leq 9(\prod_{j=1}^8 \rho_j)\Phi^4. \quad (3.7)$$

Write $x = |z_k|$, $z_k = xe^{i\theta}$, $r_k^2 = a^2 + x^2 - 2ax \cos \theta$, $\alpha = a^2 - 1 + \frac{1}{4} \sum_{l \neq k} \frac{|z_l|^2 - a^2}{r_l^2}$,

we will show

$$(\alpha + \frac{x^2 - a^2}{4r_k^2})x^{-\frac{1}{4}} \leq \alpha + \frac{1 - a^2}{4r_k^2}, \quad (3.8)$$

that is

$$\frac{x^2 - x^m}{1 - x^m} - a^2 + 4\alpha r_k^2 \leq 0.$$

Since

$$\alpha \leq a^2 - 1 + \frac{1}{4} \sum_{l \neq k} \frac{1 - a^2}{r_l^2} \leq a^2 - 1 + \frac{1}{4}(1 - a^2)\sigma - \frac{1 - a^2}{4r_k^2},$$

and $r_k \leq a + x$, it is sufficient to have the condition (3.6)

$$\frac{x^2 - 1}{(1 - x^m)(a + x)^2} + (1 - a^2)(\sigma - 4) \leq 0,$$

for $\sigma > 4$, we obtain (3.8).

Apply (3.8) eight times we have

$$\Phi \leq \frac{1}{4}(1 - a^2)(\sigma - 4).$$

Using this inequality in (3.7), the lemma follows.

Write $R_k = r_k \prod_{j=1}^8 r_j^{-\frac{1}{8}}$,

$$U^*(a) = (8 - v^*)(\frac{2 \sin \frac{\pi}{9}}{1 + a})^{-\frac{7}{4}} + (v^* - 1)(\frac{1 + a}{9^{\frac{1}{8}}})^{-2} + \{(\frac{2 \sin \frac{\pi}{9}}{1 + a})^{\frac{7}{8}(8 - v^*)}(\frac{1 + a}{9^{\frac{1}{8}}})^{v^* - 1}\}^2, \quad (3.9)$$

where

$$v^* = \min\{j \in \mathbb{Z} : j \geq \{7 \log(\frac{1+a}{2 \sin \frac{\pi}{9}})\}(\log\{\frac{(1+a)^{\frac{15}{8}}}{9^{\frac{1}{8}}(2 \sin \frac{\pi}{9})^{\frac{7}{8}}}\})^{-1}\}. \quad (3.10)$$

Lemma 3.13. If $\rho_1 > 1$, then

$$\sum_{k=1}^8 \frac{1}{R_k^2} \leq U^*(a).$$

Proof. We have $\prod_{k=1}^8 R_k = 1$. By (1.1), (1.2), and Lemma 3.2, we deduce

$$\left(\frac{2 \sin \frac{\pi}{9}}{1+a}\right)^{\frac{7}{8}} \leq R_k \leq \frac{1+a}{9^{\frac{1}{8}}}.$$

Taking $N = 8$, $C = 1$, $m = \left(\frac{2 \sin \frac{\pi}{9}}{1+a}\right)^{\frac{7}{8}}$, $M = \frac{1+a}{9^{\frac{1}{8}}}$, $c_k = R_k$ in Lemma 3.1, the lemma follows.

By (1.1), (1.2), Lemma 3.1 and Lemma 3.2, we obtain

$$\sigma \leq U(a), \quad (3.11)$$

where

$$U(a) = (8-v)(2 \sin \frac{\pi}{9})^{-2} + (v-1)(1+a)^{-2} + \{(2 \sin \frac{\pi}{9})^{8-v}(1+a)^{v-1}9^{-1}\}^2, \quad (3.12)$$

and

$$v = \min\{j \in \mathbb{Z} : j \geq \log\{9(2 \sin \frac{\pi}{9})^{-8}\}(\log\{(1+a)(2 \sin \frac{\pi}{9})^{-1}\})^{-1}\}. \quad (3.13)$$

Lemma 3.14. If $\rho_1 > 1$, $4 < \sigma \leq U(a) < \frac{64}{9}$, $a \in [0.5, 0.95]$, then

$$\frac{4U(a)}{U(a)-4} \left(\frac{8-\frac{9}{8}U(a)}{(1-a^2)^3}\right)^{\frac{1}{4}} \leq U^*(a).$$

Proof. Let $a \in [0.5, 0.95]$, by Lemma 3.10, the following inequality is sufficient for (3.6)

$$\lim_{x \rightarrow 1-0} \frac{x^2-1}{(1-x^m)(a+x)^2} + (1-a^2)(\sigma-4) \leq 0,$$

that is $\sigma \leq 4 + \frac{8}{(1-a)(1+a)^3}$, this is true for $\sigma < \frac{64}{9}$.

By Lemma 3.12, Lemma 3.13 and Lemma 3.2,

$$4^4(8-\frac{9}{8}\sigma)(1-a^2)^{-3} \leq \{(9 \prod_{j=1}^8 \rho_j)^{\frac{1}{4}}\sigma - 4(9 \prod_{j=1}^8 \rho_j)^{\frac{1}{4}}\}^4$$

$$= \left\{ \sum_{k=1}^8 \frac{1}{R_k^2} - 4(9 \prod_{j=1}^8 \rho_j)^{\frac{1}{4}} \right\}^4 \leq \{U^*(a) - 4(9 \prod_{j=1}^8 \rho_j)^{\frac{1}{4}}\}^4.$$

Using Lemma 3.12 again, we deduce

$$(8 - \frac{9}{8}\sigma)^{\frac{1}{4}}(1 - a^2)^{-\frac{3}{4}} \leq \frac{1}{4}U^*(a) - 4(8 - \frac{9}{8}\sigma)^{\frac{1}{4}}(1 - a^2)^{-\frac{3}{4}}(\sigma - 4)^{-1},$$

using (3.11), (3.12) and (3.13), the lemma follows.

Using (3.10) and (3.13), we obtain

$$v^* = 6 : a \geq 9^{\frac{3}{17}}(2 \sin \frac{\pi}{9})^{-\frac{7}{17}} - 1 = 0.723 \dots = a_2;$$

$$v^* = 7 : a < a_2, \ a \geq (\frac{9}{2 \sin \frac{\pi}{9}})^{\frac{1}{7}} - 1 = 0.445 \dots .$$

$$v = 5 : a \geq 9^{\frac{1}{5}}(2 \sin \frac{\pi}{9})^{\frac{5-8}{5}} - 1 = 0.948 \dots ;$$

$$v = 6 : a < 0.948 \dots, \ a \geq 9^{\frac{1}{6}}(2 \sin \frac{\pi}{9})^{\frac{6-8}{6}} - 1 = 0.636 \dots = a_1;$$

$$v = 7 : a < a_1, \ a \geq 0.445 \dots .$$

We divide the range of a into intervals: $[0.5195, a_1)$, $[a_1, a_2)$, $[a_2, 0.845)$.

If $\rho_1 > 1$, by (3.9) and (3.12), we get contradiction with Lemma 3.14 .

Hence, by Lemma 3.7, we obtain Theorem 2.

4 Proof of the Theorem 3

As in [12], for any function $h(z)$, write

$$M(h) = \exp(\frac{1}{2\pi} \int_0^{2\pi} \log |h(e^{i\theta})| d\theta).$$

$$\text{Given two polynomials } Q_n(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k \text{ and } R_n(z) = \sum_{k=0}^n \binom{n}{k} b_k z^k,$$

$$\text{we define the Szegő-composition polynomial } Q_n \otimes R_n = \sum_{k=0}^n \binom{n}{k} a_k b_k z^k.$$

Brujn-Springer Theorem. If $a_n b_n \neq 0$, we have

$$M(Q_n \otimes R_n) \leq M(Q_n)M(R_n). \quad (4.1)$$

Proof. See Theorem 7 of [6] and [12].

We will use the following well-known formula

$$\frac{1}{2\pi} \int_0^{2\pi} \log |r - e^{i\theta}| d\theta = \max\{0, \log |r|\}, \quad r \in \mathbb{C}. \quad (4.2)$$

Let $p(z)$ be the polynomial in the introduction, write

$$\begin{aligned}
p(z+a) &= zQ(z+a) = \sum_{k=1}^n \frac{Q^{(k-1)}(a)}{(k-1)!} z^k, \\
p'(z+a) &= \sum_{k=0}^{n-1} \frac{Q^{(k)}(a)}{k!} (k+1) z^k, \\
p(z+a) &= z \prod_{k=1}^{n-1} (z - \gamma_k), \quad \gamma_k = z_k - a, \quad r_k = |\gamma_k|, \\
p'(z+a) &= n \prod_{j=1}^{n-1} (z - \beta_j), \quad \beta_j = \zeta_j - a, \quad \rho_j = |\beta_j|.
\end{aligned}$$

We have

$$\begin{aligned}
Q(z\rho+a) &= \sum_{k=0}^{n-1} \frac{Q^{(k)}(a)}{k!} \rho^k \binom{n-1}{k}^{-1} \binom{n-1}{k} z^k \\
&= \sum_{k=0}^{n-1} \frac{Q^{(k)}(a)}{k!} (k+1) \rho^{mk} \binom{n-1}{k}^{-1} \binom{n-1}{k} z^k \otimes \sum_{k=0}^{n-1} \frac{1}{k+1} \rho^{(1-m)k} \binom{n-1}{k} z^k,
\end{aligned}$$

hence

$$Q(z\rho+a) = p'(z\rho^m+a) \otimes \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} (z\rho^{1-m})^k. \quad (4.3)$$

Lemma 4.1. We have

$$M(p'(z\rho+a)) = n\rho^{n-1} \prod_{\rho_k \geq \rho} \rho_k \rho^{-1}.$$

Proof. By (4.2) and $p'(z\rho+a) = n \prod_{k=1}^{n-1} (z\rho - \beta_k)$, we deduce

$$\begin{aligned}
M(p'(z\rho+a)) &= \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \left\{ \log n + \sum_{k=1}^{n-1} \log |\rho e^{i\theta} - \beta_k| \right\} d\theta\right) \\
&= n\rho^{n-1} \exp\left(\sum_{k=1}^{n-1} \frac{1}{2\pi} \int_0^{2\pi} \log |\rho^{-1}\beta_k - e^{i\theta}| d\theta\right) = n\rho^{n-1} \prod_{\rho_k \geq \rho} \rho_k \rho^{-1},
\end{aligned}$$

the lemma follows.

Lemma 4.2. We have

$$M\left(\sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} (z\rho^{1-m})^k\right) = \frac{1}{n} \rho^{(1-m)(n-1)} \prod_{\rho^{m-1} 2 \sin \frac{\pi k}{n} \geq 1} \rho^{m-1} 2 \sin \frac{\pi k}{n}.$$

Proof. Since

$$\sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} (z\rho^{1-m})^k = \frac{1}{n} \prod_{k=1}^{n-1} (z\rho^{1-m} - (e^{2\pi i \frac{k}{n}} - 1)),$$

by (4.2), we obtain

$$M\left(\sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} (z\rho^{1-m})^k\right) = \frac{1}{n} \rho^{(1-m)(n-1)} \prod_{\rho^{m-1} 2 \sin \frac{\pi k}{n} \geq 1} \rho^{m-1} 2 \sin \frac{\pi k}{n},$$

the lemma follows.

But

$$Q(z\rho + a) = \prod_{k=1}^{n-1} (z\rho - \gamma_k), \quad M(Q(z\rho + a)) = \rho^{n-1} \prod_{r_k \geq \rho} r_k \rho^{-1}, \quad (4.4)$$

by Lemma 4.1, Lemma 4.2, (4.1), (4.3) and (4.4), Theorem 3 follows.

Remark 4.1. It is possible to obtain new results on the Sendov conjecture by combining Theorem 3 with Lemma 3.1.

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